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Exact solitons on non-commutative tori

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Abstract: We construct exact solitons on non-commutative tori for the type of actions arising from open string field theory. Given any projector that describes an extremum of the tachyon potential, we interpret the remaining gauge degrees of freedom as a gauge theory on the projective module determined by the tachyon. Whenever this module admits a constant curvature connection, it solves exactly the equations of motion of the effective string field theory. We describe in detail such a construction on the non-commutative tori. Whereas our exact solution relies on the coupling to a gauge theory, we comment on the construction of approximate solutions in the absence of gauge fields.

Keywords: D-branes, Non-Commutative Geometry.

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Contents

1. Introduction

Since the original conjectures by Sen [1] on the emergence of D-branes as solitons in open string field theory, there has been much progress in many directions. In particular it was observed [2] that introducing large B-field enables one to describe D-branes as non-commutative solitons [3, 4]. Starting with the open string field theory and integrating out all the massive and unphysical degrees of freedom one obtains some unknown effective action with certain conjectured properties. These are just enough to construct lower-dimensional D-branes as solitons in the tachyon field. Later on it was realized [5] that the assumption of large noncommutativity or B-field is not actually necessary. Turning on appropriate gauge field which sets to zero all the covariant derivatives of the tachyon, one can construct exact non-commutative solitons. For a recent review see [6].

Most of the work on the non-commutative solitons has been done so far for the non-commutative Moyal plane. Its nice property is that the algebra of functions on the plane with the Moyal star product is isomorphic to the algebra of compact operators on a separable Hilbert space through the Weyl quantization formula. This allows one to use familiar methods of the quantum mechanical harmonic oscillator. The next to simple and more realistic problem of solitons on the non-commutative torus studied in [7]–[10] requires however more involved mathematics of non-commutative geometry.

The aim of the present paper is to find exact soliton solutions on the torus for the type of action arising from string field theory. This problem has been recently addressed in a pioneering work [10]. Our paper completes the latter in the sense that it provides a general method for constructing solutions of the problem they raised.

The effective action contains two fields: the tachyon field and the gauge field which are coupled through a covariant derivative. The tachyon equation of motion can be satisfied by picking up any projector (so that the potential energy is minimal) and finding a compatible connection (thus the covariant derivative of the tachyon vanishes). It is a fortunate fact that for a given projector the space of all compatible connections turns out to be remarkably simple. Besides, among all the compatible connections one can select those that solve exactly the remaining equations of motion of the gauge theory. The latter are just the connections whose curvature is expressed as a linear combination of projectors entering in a decomposition of the tachyon field. They are related to the so-called constant curvature connections that have been extensively studied in the mathematical literature [11]–[15]. We provide explicitly such a link in the simplest possible situation, but our method is general and yields exact solutions in a much broader context.

The paper is organized as follows. In section 2 we first review the non-commutative effective field theory description of the tachyon and gauge fields. We show in general how one can satisfy all the equations of motion and determine the space of all compatible connections. In section 3 we use the bimodule technique to provide an explicit solution of the gauge theory. In section 4 we show that the solutions we have found have the right tension to be interpreted as D-branes and we demonstrate the independence on the choice of the Seiberg-Witten two form Φ used in the description. Section 5 is devoted to an approximate solution in the absence of gauge fields. This leads us to a projector, known as Boca's projector, which is related to the particular solution found on the Moyal plane. The relation between the GMS projector and Boca's projector in the large torus limit is further studied in appendix A.

2. Effective description of string field theory

We start by considering the open bosonic string theory propagating in the closed string background of the form $\mathcal{M} \times \mathbb{T}^d$ where $\mathcal{M}$ is arbitrary 26-d-dimensional manifold. The effect of turning on a constant B-field (for a review see [16]) along a flat submanifold, in this case the torus $\mathbb{T}^d$, is neatly described by replacing ordinary products by the non-commutative star products defined as

$$f * g = f\, e^{\frac{i}{2}\theta^{ij}_{\mathrm{Moyal}} \overleftarrow{\partial}_i \overrightarrow{\partial}_j} g\,, \tag{2.1}$$

where $\theta^{ij}_{\mathrm{Moyal}}$ is an antisymmetric real matrix.

One also has to replace the ordinary closed string metric g_{ij} and coupling constant g_s by effective open string parameters G_{ij} and G_s. All these effective parameters are related to the closed ones through

$$\frac{1}{G+2\pi\alpha'\Phi} = -\frac{\theta_{\mathrm{Moyal}}}{2\pi\alpha'} + \frac{1}{g+2\pi\alpha' B}\,,$$
$$G_s = g_s\left(\frac{\det(G+2\pi\alpha'\Phi)}{\det(g+2\pi\alpha' B)}\right)^{1/2}, \tag{2.2}$$

where Φ is the Seiberg-Witten two form [16, 17] which always appears in the combination with the gauge field strength[1] $iF+\Phi$. Its appearance is related to a freedom in the regularization of the worldsheet theory.

All the fields are regarded as functions of the commutative coordinates x^a and are valued in the non-commutative algebra $\mathcal{A}_\theta$. In more concrete terms, this algebra is generated by functions $U_i = e^{i\frac{x^i}{R}}$ having the commutation relations

$$U_i * U_j = e^{-\frac{i}{R^2}\theta^{ij}_{\mathrm{Moyal}}} U_j * U_j\,. \tag{2.3}$$

For simplicity we take all radii of the torus equal to R, but all what follows is easily adapted to any constant metric.

To make contact with the standard conventions for the non-commutative torus we shall omit the stars and set

$$2\pi\theta^{ij} = -\frac{\theta^{ij}_{\mathrm{Moyal}}}{R^2}\,. \tag{2.4}$$

The algebra is spanned by $U_{\vec{n}} = \prod_{i=1}^d U_i^{n_i}$ where $\vec{n}\in\mathbb{Z}^d$.

The standard partial derivatives with respect to the coordinates x^i are derivations of the algebra $\mathcal{A}_\theta$, i.e. they also satisfy the Leibniz rule for the star product.

The ordinary integral over the torus yields a trace on the algebra, i.e. two elements of $\mathcal{A}_\theta$ commute under the integral. In accordance with the standard trace that can be found in the mathematical literature, we choose to normalize it as follows,

$$\frac{1}{(2\pi R)^d}\int_{\mathbb{T}^d}\sum_{\vec{n}\in\mathbb{Z}^d} a_{\vec{n}} U_{\vec{n}} = a_{\vec{0}}\,. \tag{2.5}$$

Accordingly, this trace will be referred to as the "normalized integral". We refrain from calling it the trace, since one also considers matrices with entries in $\mathcal{A}_\theta$. On the algebra $M_N(\mathcal{A}_\theta)$, one introduces the ordinary linear form Tr from $M_N(\mathcal{A}_\theta)$ to $\mathcal{A}_\theta$ as the sum of the diagonal elements. It is not a trace since the property $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ is lost when the algebra is not commutative. However, this property is true after integration.

[1] We are using geometric conventions where A and F are antihermitean. The relation with [16] is $A^{SW} = iA^{\mathrm{here}}$.

The effective action is obtained by integrating out all the fields from open string field theory except the tachyon T and the gauge field A_μ. It takes the general form

$$
\begin{aligned}
S = \frac{c}{G_s}\int_{\mathcal{M}} d^{26-d}x \int_{\mathbb{T}^d} \sqrt{\det G}\ \mathrm{Tr}\,\Big[&\frac{1}{2} f(T^2-1) G^{\mu\nu} D_\mu T D_\nu T - V(T^2-1) - \\
&-\frac{1}{4}(2\pi\alpha')^2 h(T^2-1)(iF_{\mu\nu}+\Phi_{\mu\nu})(iF^{\mu\nu}+\Phi^{\mu\nu}) + \cdots\Big],
\end{aligned}
\tag{2.6}
$$

where $\mu,\nu = 1,2,\ldots,26$ and $c = g_s T_{25}$ is a B independent constant. The tachyon field T is a $\mathcal{A}_\theta$-valued function on $\mathcal{M}$ and transforms in the adjoint representation. The gauge field A_μ is an antihermitean matrix of functions with values in $\mathcal{A}_\theta$ and its curvature is defined as usual. This action is invariant under the standard non-commutative gauge transformations.

The explicit form of the effective action is not completely known. It may contain higher order terms that are represented in (2.6) by dots. The latter are constructed using products of higher order covariant derivatives of the tachyon field and of the curvature tensor. Fortunately their explicit form is not necessary in order to apply the method we shall describe below.

Furthermore, the functions f, h, V are only required to satisfy certain conjectured properties. The following ones are not intended to be the complete list but merely those we shall need in what follows,

$$
\begin{aligned}
V(0) &= 0\,, & h(0) &= 0\,, & f(0) &= 0\,,\\
V(-1) &= 1\,, & h(-1) &= 1\,, & &\\
V'(0) &= 0\,, & h'(0) &= 0\,. & &
\end{aligned}
\tag{2.7}
$$

The tachyon potential is normalized in such a way that at the closed string vacuum $T = 1$ it has a minimum equal to zero and at $T = 0$ it has a local maximum equal to 1. The conditions for $h(0)$ and $f(0)$ reflect Sen's conjecture that at the closed string vacuum all kinetic terms of physical excitations do vanish. This conjecture has been recently tested numerically [19, 20] and was also a starting point of [21, 22]. The condition on $h(-1)$ is just a normalization. What is less clear is the physical meaning of $h'(0) = 0$. It follows nevertheless from the Dirac-Born-Infeld extension [23, 24] (see also [25, 26]) which implies $h = V$. Among the higher derivative terms, those constructed solely from the curvature F and the tachyon T (without derivatives) will be quite important later. Because of the Dirac-Born-Infeld extension, their coefficients $h_{(n)}$ also satisfy $h_{(n)}(0) = 0$ and $h'_{(n)}(0) = 0$.

The action (2.6) leads to two equations of motion obtained by variation with respect to tachyon and gauge fields. They are obtained by expanding the functions f, g and V in power series and collecting the coefficients of δT and δA_μ after repeated integrations by parts and cyclic permutations of the fields. For the sake of brevity, we do not give their explicit form here. Indeed, they are quite cumbersome due to

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the noncommutativity between the fields and their variations. Instead, we list a few conditions on the fields that are sufficient in order to solve the equations of motion

$$\begin{aligned} V'(T^2-1)T &= 0\,, \\ D_\mu T &= 0\,, \\ D_\alpha F_{\mu\nu} &= 0\,, \\ \int_{\mathbb{T}^d}\left[\delta h(T^2-1)(iF_{\mu\nu}+\Phi_{\mu\nu})(iF^{\mu\nu}+\Phi^{\mu\nu})\right] &= 0\,, \\ &\cdots\,, \\ \int_{\mathbb{T}^d}\left[\delta h_{(n)}(T^2-1)(iF+\Phi)^{2n}\right] &= 0\,, \\ &\cdots\,. \end{aligned} \tag{2.8}$$

These conditions imply the equations of motion whatever the higher order terms are. Indeed, the first one ensures that the contribution to δT arising from the tachyonic potential is identically zero. After integration by parts, the second and the third one imply that any contribution containing covariant derivatives of the fields vanish. Finally the last ones are the remaining contributions from the monomials in the curvature and the tachyon field without any covariant derivative.

Thanks to the condition $V'(0) = 0$ the first equation is solved by taking T to be any projector which we write as $T = 1 - P$. Therefore the second equation becomes

$$\partial_\mu P + A_\mu P - PA_\mu = 0\,. \tag{2.9}$$

For later purposes it is convenient to solve a slightly more complicated equation. Suppose that we are given N hermitean elements $P_1, \ldots, P_n$ of the algebra $\mathcal{A}_\theta$ such that

$$P_iP_j = \delta_{ij}P_i\,, \qquad \text{and} \qquad \sum_{1\leq i\leq n} P_i = 1\,. \tag{2.10}$$

This implies that they form a set of mutually orthogonal projectors whose sum is the identity. In particular, if we have a single projector P, then P and $1 - P$ form such a set.

The generalization of the single equation (2.9) is the set of equations

$$dP_i + AP_i - P_iA = 0 \tag{2.11}$$

whose general solution is given by

$$A = \sum_{1\leq i\leq n} A_i + P_i dP_i\,, \tag{2.12}$$

where A_i are arbitrary hermitean elements of $\mathcal{A}_\theta$ such that $P_iA_iP_i = A_i$. The solution is easily obtained by decomposing A into its "matrix elements"

$$A = \sum_{1\leq i,j\leq n} P_iAP_j\,. \tag{2.13}$$

Whereas the diagonal terms P_iAP_i are arbitrary, the off-diagonal ones are completely fixed by (2.11). In these formulae, we have used coordinate the free notations of differential geometry. The reader should bear in mind that these are just a convenient notation that avoids a proliferation of space-time indices and they are defined in the non-commutative case by the same formulae as in the commutative case.

Let us suppose that P has been written as

$$P = \sum_{1\leq i\leq m} P_i \tag{2.14}$$

and that

$$1 - P = \sum_{m+1\leq i\leq n} P_i\,, \tag{2.15}$$

where $P_1, \ldots, P_n$ is a system of mutually orthogonal projectors whose sum is one. Such a system always exists (take for instance P and $1 - P$), but considering a more general solution will be useful in the sequel.

Then the gauge field A we have constructed satisfies (2.11). Summing over i from 1 to M, one concludes that it also satisfies (2.9). Accordingly, we have solved the equation $DT = 0$ and we are left with the last equations involving the curvature.

The curvature $F = dA + A^2$ is easily computed using

$$dP_iP_j + P_idP_j = \delta_{ij}dP_i\,. \tag{2.16}$$

It turns out to be diagonal

$$F = \sum_{1\leq i\leq n} \left(P_idP_idP_iP_i + P_idA_iP_i + A_i^2\right). \tag{2.17}$$

This fact is quite crucial since it allows us to satisfy identically the last set of equations in (2.8) for the tachyon. To see that, let us expand h in power series, bearing in mind that $h(0) = h'(0) = 0$,

$$h(T^2 - 1) = \sum_{n\geq 2} \frac{1}{n!}h^{(n)}(0)\left(T^2 - 1\right)^n, \tag{2.18}$$

so that its variation is easily computed by varying all monomials. The variation of a monomial of degree n gives rise to $2n$ terms

$$\delta\left(T^2 - 1\right)^n = \sum_{0\leq k\leq n-1} \left(T^2 - 1\right)^k (T\delta T + \delta TT)\left(T^2 - 1\right)^{n-1-k}. \tag{2.19}$$

Although δT is completely arbitrary, when $T = 1 - P$ is a projector, this formula simplifies into

$$\delta\left(T^2 - 1\right)^n = (-1)^n \left(T\delta T\left(T^2 - 1\right) + \left(T^2 - 1\right)\delta TT\right) \tag{2.20}$$

so that it vanishes identically when multiplied with a diagonal quantity and integrated. The same analysis remains true for the higher order terms involving $h_{(n)}$ so that we can conclude that the last set of equations in (2.8) is satisfied.

We are thus left with the only remaining equation $D_\alpha F_{\mu\nu} = 0$. A further simplification occurs because of $D_\alpha P = 0$. This allows to solve the last equation as follows. Suppose we can find gauge fields A_i (with $P_i A_i P_i = A_i$) such that

$$P_i dP_i dP_i + P_i dA_i P_i + A_i^2 = \lambda_i P_i \tag{2.21}$$

with $\lambda_i \in \mathbb{C}$ being numbers. Then the compatibility condition (2.11) implies

$$D_\alpha F = \sum_{1 \leq i \leq n} \lambda_i D_\alpha P_i = 0\,, \tag{2.22}$$

which ensures that the full set of equations of motion are satisfied, whatever the higher order terms are. The existence of such gauge fields as well as some explicit constructions will be given in the next section.

3. Constant curvature connections

To solve (2.21) we have to make contact with some available results in the mathematical literature. In fact, it is equivalent to the existence of a constant curvature connections on the projective module determined by P_i. On the non-commutative tori this has been extensively studied, yet formulated in a rather different way, using functional analytic methods. This way of presenting projective modules over non-commutative tori and the associated constant curvature connections have been proposed by A. Connes more than twenty years ago [11].

To proceed, let us first recall that a projector P in $M_N(\mathcal{A}_\theta)$ determines a finitely generated projective module $\mathcal{E} = P\mathcal{A}_\theta^N$. Alternatively, one can consider $\mathcal{E}$ as the subspace of elements ξ of $\mathcal{A}_\theta^N$ that fulfill $P\xi = \xi$. This is naturally a right $\mathcal{A}_\theta$ module since it is stable by multiplication by elements of $\mathcal{A}_\theta$ on the right.

Projective modules are fundamental objects in non-commutative geometry since they provide a generalization of vector bundles. Moreover, one can define the analogue of a covariant derivative and its curvature for these modules. In fact, a covariant derivative ∇_μ associated to the partial derivative ∂_μ is just a linear map from $\mathcal{E}$ to itself satisfying the Leibniz rule

$$\nabla_\mu(P\xi f) = \nabla_\mu(P\xi)f + P\xi\partial_\mu f \tag{3.1}$$

for any $P\xi \in \mathcal{E}$ and $f \in \mathcal{A}_\theta$. It is not difficult to construct a covariant derivative just by taking the ordinary partial derivative and projecting the result onto $\mathcal{E}$ using P. This defines the covariant derivative ∇_μ^0 by

$$\nabla_\mu^0(\xi) = P\partial_\mu(P\xi) \tag{3.2}$$

for any $P\xi \in \mathcal{E}$.

To the module $\mathcal{E}$ we also associate its algebra of endomorphisms $\mathrm{End}(\mathcal{E})$ which is the algebra of linear maps from $\mathcal{E}$ to itself that commute with the right action of $\mathcal{A}_\theta$. It is isomorphic to the algebra of matrices of the form $PAP \in M_N(\mathcal{A}_\theta)$, or equivalently, that satisfy $PAP = A$, the sum and product being the ordinary operations on matrices but the unit is the projector P instead of 1.

It can be easily shown that all connections are of the form

$$\nabla_\mu = \nabla^0_\mu + A_\mu \,, \tag{3.3}$$

where A_μ is a matrix satisfying $PA_\mu P = A_\mu$. In a language more familiar to physicists A_μ may be identified with the gauge field. The gauge transformations are the unitary elements of $\mathrm{End}(\mathcal{A}_\theta)$, that is those $g \in \mathrm{End}(\mathcal{E})$ that fulfill $gg^* = g^*g = P$. Note that one usually imposes that A_μ be antihermitean and it transforms in the ordinary way under gauge transformations.

The curvature tensor is defined as the commutator of the covariant derivatives,

$$F_{\mu\nu} = [\nabla_\mu, \nabla_\nu] \,. \tag{3.4}$$

Unlike the covariant derivatives, it commutes with the right action of $\mathcal{A}_\theta$, so that it may be identified with an element of $\mathrm{End}(\mathcal{A}_\theta)$.

If we compute the curvature of the connection defined in (3.3), we get

$$F_{\mu\nu} = P[\partial_\mu P, \partial_\nu P]P + P\partial_\mu A_\nu P - P\partial_\nu A_\mu P + [A_\mu, A_\nu] \tag{3.5}$$

which has exactly the form of the diagonal elements of (2.17). Since P is the identity of $\mathrm{End}(\mathcal{E})$, gauge fields whose curvature is proportional to P are known in the mathematical literature as "constant curvature connections".

Shifting back to the notations we have been using in the previous section, let us consider our projector P corresponding to the tachyon field. This projector determines a projective module, but it has no reason, *a priori*, to admit a constant curvature connection. However, it is known that, in the irrational case (i.e. when at least one of the entries of the matrix θ_{ij} is irrational, [13, 15]), any projective module is isomorphic to a direct sum of projective modules that admit a constant curvature connection. Unfortunately, this decomposition and the associated constant curvature connections are not built using projectors. Thus our last task is to make the connection between the two points of view as explicit as possible.

Let $S : P\mathcal{A}_\theta^N \to \mathcal{E}$ be the isomorphism of $P\mathcal{A}_\theta^N$ onto the direct sum

$$\mathcal{E} = \sum_{1\leq i\leq m} \mathcal{E}_i \,, \tag{3.6}$$

where the $\mathcal{E}_i$ are projective modules that admit a constant curvature connection. This isomorphism induces an isomorphism between the associated algebra of endomorphisms so that P is mapped to the identity endomorphism $\mathrm{Id}_\mathcal{E}$ of $\mathcal{E}$.

Consider π_i the linear endomorphism of $\mathcal{E}$ that is the identity on $\mathcal{E}_i$ and that vanishes on the other factors. Obviously one has

$$\pi_i \pi_j = \delta_{ij} \pi_i \,, \qquad \text{and} \qquad \sum_{1 \leq i \leq m} \pi_i = \mathrm{Id}_{\mathcal{E}} \,. \tag{3.7}$$

Accordingly, the inverse images $P_i = S^{-1}(\pi_i)$ determine a system of m projectors that satisfy (2.10). One can perform the same construction for $1 - P$ to add $n - m$ projectors to complete the system of n projectors involved in (2.10).

Each of the n projective modules $\mathcal{E}_i$ admits a constant curvature connection ∇^i_μ. Using the isomorphism S, this constant curvature connection induces a constant curvature connection on the projective module $P_i \mathcal{A}_\theta^N$ associated to P_i which in turn yields a solution to our equation (2.21), and finally a solution to the full equation of motion.

In the two-dimensional case, this isomorphism can be described quite explicitly using the techniques developped in [11]. Let us fix an irrational number $\theta \in]0,1[$ so that the algebra $\mathcal{A}_\theta$ is generated by two unitary elements U_1 and U_2 such that

$$U_1 U_2 = e^{2i\pi\theta} U_2 U_1 \,. \tag{3.8}$$

θ is the dimensionless parameter related to the ordinary Moyal deformation parameter through $\theta_{\mathrm{Moyal}} = -2\pi\theta R^2$, R being the radius of the torus.

It is known [18] that gauge equivalence classes of projectors in $\mathcal{A}_\theta$, in the two-dimensional case and when θ is irrational, are parametrized by the values of the normalized integral. More precisely, if P is a projector, then there are two integers p and q such that

$$\frac{1}{(2\pi R)^2} \int_{\mathbb{T}^2} P = p + q\theta \,. \tag{3.9}$$

Obviously, if two projectors are gauge equivalent, they yield the same integral, thus the same numbers p and q. Conversely, given any two integers p and q such that $p + q\theta \in [0,1]$, then there is a projector in $\mathcal{A}_\theta$ such that (3.9) holds and any two such projectors are equivalent. In more physical terms, this means that p and q parametrize our vacua.

Let us now focus, for simplicity, on the case $p = 0$ and $q = 1$. Thus we are working within the gauge equivalence class of the Powers-Rieffel projector, which is the simplest possible example of a non-trivial projector in $\mathcal{A}_\theta$. We denote by P a projector within this class, $P\mathcal{A}_\theta$ is then a right $\mathcal{A}_\theta$-module ($\mathcal{A}_\theta$ acting by multiplication) which can be equivalently described as follows [11].

Let $S(\mathbb{R})$ be the space of complex valued function on $\mathbb{R}$ which decrease fast at infinity. We define a right action of $\mathcal{A}_\theta$ by

$$(\psi U_1)(t) = \psi(t + \theta) \,, \qquad (\psi U_2)(t) = e^{-2i\pi t} \psi(t) \,, \tag{3.10}$$

for any function ψ. Accordingly, this turns $S(\mathbb{R})$ into a right module which is in fact finitely generated and projective.

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Covariant derivatives ∇_1 and ∇_2 are defined by

$$\begin{aligned}\nabla_1\psi(t) &= -\frac{i}{\theta R}t\psi(t)\,,\\ \nabla_2\psi(t) &= \frac{1}{2\pi R}\frac{d}{dt}\psi(t)\,,\end{aligned} \tag{3.11}$$

which satisfy the Leibniz rule with respect to the right action of $\mathcal{A}_\theta$. The curvature of this connection is constant,

$$F_{12} = [\nabla_1, \nabla_2] = -\frac{i}{\theta_{\mathrm{Moyal}}}\,, \tag{3.12}$$

and we shall comment on the physical meaning of this value in the next section.

To describe explicitly the isomorphism between the projective module determined by the projector P and $S(\mathbb{R})$, as well as the pull back of the connection, it is convenient to use the so-called "bimodule construction".

In general terms, suppose that we are given two algebras $\mathcal{A}$ and $\mathcal{B}$ and a $(\mathcal{B}, \mathcal{A})$-bimodule $\mathcal{M}$. This means that $\mathcal{M}$ is a vector space equipped with a left action of $\mathcal{B}$ and a right action of $\mathcal{A}$ and that these two actions commute, i.e.

$$(b\psi)a = b(\psi a) \tag{3.13}$$

for any $\psi \in \mathcal{M}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Besides, let us assume that $\mathcal{M}$ comes equipped with two scalar products, $(\cdot,\cdot)_\mathcal{A}$ and $(\cdot,\cdot)_\mathcal{B}$. The first one takes its values in $\mathcal{A}$ and is $\mathcal{A}$-linear whereas the second one takes its values in $\mathcal{B}$ and is $\mathcal{B}$-linear.

We finally assume that they are compatible,

$$(\psi, \xi)_\mathcal{B}\zeta = \psi(\xi, \zeta)_\mathcal{A} \tag{3.14}$$

for any elements ψ, ξ and ζ of $\mathcal{M}$. Then one easily shows that if

$$(\psi, \psi)_\mathcal{B} = 1\,, \tag{3.15}$$

then

$$P = (\psi, \psi)_\mathcal{A} \tag{3.16}$$

is a projector in $\mathcal{A}$. Indeed,

$$\begin{aligned}P^2 &= (\psi, \psi)_\mathcal{A}(\psi, \psi)_\mathcal{A}\\ &= (\psi, \psi(\psi, \psi)_\mathcal{A})_\mathcal{A}\\ &= (\psi, (\psi, \psi)_\mathcal{B}\psi)_\mathcal{A}\\ &= (\psi, \psi)_\mathcal{A}\,.\end{aligned} \tag{3.17}$$

The use of Morita equivalence to construct projectors has been first proposed in [18] and in fact it lead Rieffel to the discovery of his projector. It generalizes the partial isometry introduced in [3] which cannot be applied to the non-commutative torus. Indeed, the normalized trace of any projector obtained by a partial isometry should be one, but then the projector is the identity.

Let us now fix $\mathcal{A} = \mathcal{A}_\theta$, $\mathcal{B} = \mathcal{A}_{-1/\theta}$ and $\mathcal{M} = S(\mathbb{R})$. The left action of $\mathcal{A}_{-1/\theta}$ is

$$\begin{aligned} V_1 f(t) &= e^{2i\pi t/\theta} f(t)\,, \\ V_2 f(t) &= f(t+1) \end{aligned} \tag{3.18}$$

and the two scalar products are defined as

$$\begin{aligned} (\psi,\chi)_{\mathcal{A}} &= \theta \sum_{m_1,m_2\in\mathbb{Z}^2} \int_{\mathbb{R}} dt \overline{\psi(t+m_1\theta)}\chi(t) e^{2i\pi m_2 t} U_1^{m_1} U_2^{m_2}\,, \\ (\psi,\chi)_{\mathcal{B}} &= \sum_{n_1,n_2\in\mathbb{Z}^2} \int_{\mathbb{R}} dt \psi(t-n_1)\overline{\chi(t)} e^{\frac{2i\pi n_2 t}{\theta}} V_1^{n_1} V_2^{n_2}\,. \end{aligned} \tag{3.19}$$

The compatibility condition follows from Poisson resummation. Note that these formulas can be extended to a much broader context [10, 13].

Besides, it is known that there exists an element ψ_0 in $S(\mathbb{R})$ such that $(\psi_0,\psi_0)_{\mathcal{B}} = 1$ [18]. The construction of such a ψ_0 is actually a non-trivial task and it is this condition that is at the origin of the strange choice of functions involved in the Powers-Rieffel projector. Thanks to the bimodule technique, we end up with a projector $P_0 = (\psi_0,\psi_0)_{\mathcal{A}}$ whose trace is θ.

We know that all projectors of trace θ in $\mathcal{A}_\theta$ are gauge equivalent. Thus the projector P we started with is related to P_0 by

$$P = u P_0 u^*\,, \tag{3.20}$$

where $u \in \mathcal{A}$ is unitary, i.e. $uu^* = u^*u = 1$. This also implies that

$$P = (u\psi_0, u\psi_0)_{\mathcal{A}}\,. \tag{3.21}$$

Writing $\psi = u\psi_0$, this allows us to define a map S from $P\mathcal{A}_\theta$ to $S(\mathbb{R})$ by

$$S(Pa) = \psi P a \tag{3.22}$$

for any $Pa \in \mathcal{A}_\theta$. It is a right $\mathcal{A}_\theta$-module homomorphism which turns out to be invertible. Indeed, its inverse is given by

$$S^{-1}(\xi) = (\psi,\xi)_{\mathcal{A}}\,. \tag{3.23}$$

Therefore, it establishes a right $\mathcal{A}_\theta$-module homomorphism between $P\mathcal{A}_\theta$ and $S(\mathbb{R})$.

Using the covariant derivatives on $S(\mathbb{R})$ defined in (3.11) we define a covariant derivative $S^{-1}\nabla_\mu S$ on the algebra $\mathcal{A}$. In terms of gauge fields, we have

$$\begin{aligned} S^{-1}\nabla_\mu S(Pa) &= (\psi, \nabla_\mu(\psi P a))_{\mathcal{A}} \\ &= (\psi,\nabla_\mu\psi)_{\mathcal{A}} Pa + (\psi,\psi)_{\mathcal{A}} \partial(Pa) \\ &= A_\mu Pa + P\partial_\mu(Pa)\,, \end{aligned} \tag{3.24}$$

with

$$A_\mu = (\psi, \nabla_\mu \psi)_{\mathcal{A}} (\psi, \psi)_{\mathcal{A}} \,. \tag{3.25}$$

It can be checked by direct computation, using the Leibniz rule to derive the scalar products, that the curvature associated to A_μ is proportional to P,

$$F_{\mu\nu} = P[\partial_\mu P, \partial_\nu P]P + P\partial_\mu A_\nu P - P\partial_\nu A_\mu P + [A_\mu, A_\nu] = -\frac{i}{\theta_{\mathrm{Moyal}}} P \,. \tag{3.26}$$

Obviously this construction can be generalized to projectors with other traces and in higher dimensions. One can thus construct, with a high level of explicitness, gauge fields compatible with given projectors that extremize the effective action (2.6). Note that the above formula (3.25) has already appeared in the physical literature in [27] in the study of non-commutative instantons.

Let us end this section by a comment on the gauge theoretical aspect of the procedure we followed. First we have found a non-trivial extremum of the potential which is just the projector $T_P = 1 - P$. The latter induces spontaneous symmetry breaking since it is not invariant under the full gauge group G. The subgroup H_P of all unitary elements u such that $uT_P u^* = T_P$ is the unbroken subgroup and the compatibility condition (2.11) just means that we are only considering a gauge theory with the little group H_P as gauge group. From this interpretation, it follows that all projectors are equally good. Indeed, trading P for uPu^* is just a gauge transformation and it leaves the physics invariant. Our truly independent solutions are in fact equivalence classes of projectors. On the non-commutative tori, the latter are known to form a discrete set.

4. D-brane tension

To verify that the soliton solutions we have found correspond to D-branes, we can calculate their tension as in [2, 5, 6]. We shall specialize to the solitons on the non-commutative two torus for which we were able to make everything rather explicit. As they are localized in two directions they should represent codimension two branes, i.e. D23-branes.

Let us take first for the Seiberg-Witten parameter Φ the simplest choice $\Phi = -B = -1/\theta_{\mathrm{Moyal}}$ which sets to zero all the terms in the action containing $iF + \Phi$. The only contribution to the tension then comes only from the potential term. Using

$$\frac{\sqrt{\det G}}{G_s} = \frac{2\pi\alpha'}{g_s |\theta|_{\mathrm{Moyal}}} \tag{4.1}$$

we get for the action evaluated on our solution

$$S = -\frac{2\pi\alpha' c}{g_s |\theta|_{\mathrm{Moyal}}} \int_{\mathbb{T}^d} P \int_{\mathcal{M}} d^{26-d}x \sqrt{\det g_{\mathcal{M}}} \,, \tag{4.2}$$

since

$$V(T^2-1) = V(-P) = V(0) + (V(-1) - V(0))P = P\,. \tag{4.3}$$

As we have discussed above the projectors on the algebra $\mathcal{A}_\theta$ are classified according to their trace which belongs to $(\mathbb{Z} + \theta\mathbb{Z}) \cap [0,1]$. For a projector P of normalized trace θm we get the tension

$$T = \frac{2\pi\alpha' c}{g_s|\theta|_{\mathrm{Moyal}}}(2\pi R)^2\theta m = (2\pi)^2\alpha' m T_{25} = m T_{23} \tag{4.4}$$

which identifies our soliton to be describing m D23-branes. Curiously the mathematical theorem tells us that the soliton number cannot exceed $1/\theta$. Physically it is welcome since the total energy of the D23-branes should be smaller than the energy of the original unstable D25-brane.

We may ask what would happen if we had chosen different value for the parameter Φ. Would we end up with the right tension? The value we took before was special in the sense that all terms containing the curvature vanished. If we take different value for Φ we have to know all terms in the action which do not contain derivatives. Fortunately these are precisely those terms provided by the Dirac-Born-Infeld action [23, 24]

$$S = \frac{c}{G_s}\int_{\mathcal{M}} d^{26-d}x \int_{\mathbb{T}^d}\left[-V(T^2-1)\sqrt{\det(G + 2\pi\alpha'(iF + \Phi))} + \cdots\right]. \tag{4.5}$$

Clearly the $1-P$ part of F will not contribute. We can use formula

$$\sqrt{\det(a+bP)} = \sqrt{\det a}\,(1-P) + \sqrt{\det(a+b)}\,P\,. \tag{4.6}$$

It is then simple exercise to check using (2.2) that

$$\frac{\sqrt{\det(G + 2\pi\alpha'(\theta^{-1}_{\mathrm{Moyal}} + \Phi))}}{G_s} = \frac{\sqrt{|\det\theta^{-1}_{\mathrm{Moyal}}|}}{g_s} \tag{4.7}$$

is actually Φ independent and when we evaluate the action we get the same tensions as before.

5. Approximate solutions without gauge fields and selfdual projectors

In the previous sections we were solving exactly the equations of motion by turning on an appropriate gauge field. Obviously one may ask whether it is possible to satisfy the equations without turning on the gauge field at all. This should indeed be possible as suggest the results in CSFT [28, 29, 30] and BSFT [31, 32]. In the effective field theory approach we are pursuing here we are then limited only to approximate solutions neglecting higher derivatives.

Natural procedure is to first exclude all the derivative terms and search for the minima of the potential. As is known [3] non-trivial solutions of $V'(\phi) = 0$ can be constructed using projectors. The next step is to take into account the kinetic term, which reduces to

$$S_{\rm kin}[P] = \frac{1}{\lambda^2} \int P\partial_\mu P\partial_\mu P \tag{5.1}$$

when evaluated on a projector P. Note that we have summarized all relevant information on the function f and the effective couplings into the "coupling constant" λ. Whereas all projectors are ground states of the potential, only a few of them will minimize the correction given by the kinetic term. Let us thus try to find the extrema of (5.1) on the space of all projectors.

This problem has been studied in [33]. Let us first derive the equation of motion. If P is a projector, then $P + \delta P$ is a projector (at the first order) iff $\delta P = [\delta a, P]$ with $\delta a \in \mathcal{A}$. Accordingly, the equation of motion are

$$P\,\Delta P - \Delta P\,P = 0\,, \tag{5.2}$$

where Δ is the standard laplacian. It is worthwhile to notice that this equation takes a particular form because we search for the extrema of (5.1) on the space of projectors. If we were instead working on the space of all scalar fields, the equations of motion would be completely different and would include also a non-trivial dependence on the function f appearing in the effective action (2.6). To our knowledge it is not clear that the extrema of the full action (in the absence of gauge field and neglecting the higher order derivatives of the tachyon) can be obtained by first searching for the zeros of the potential and then minimizing the kinetic term on these zeros.

However, the problem of minimizing the kinetic term on the space of projectors is an interesting question since it involves some topologically stable solutions in two dimension. In fact this action admits a topological bound

$$\frac{1}{\lambda^2} \int P\partial_\mu P\partial_\mu P \geq \frac{1}{\lambda^2} \left| i\epsilon_{\mu\nu} \int P\partial_\mu P\partial_\nu P \right| . \tag{5.3}$$

This is easily derived from the relation

$$\int (\partial PP)^* \partial PP = \int P\partial_\mu P\partial_\mu P - i\epsilon_{\mu\nu} \int P\partial_\mu P\partial_\nu P\,, \tag{5.4}$$

where $\partial = \partial_1 - i\partial_2$ and from the corresponding equality involving $\overline{\partial}$.

This inequality (5.3) is similar to the inequality arising in four-dimensional Yang-Mills theory and is interpreted as follows. The space of projectors is not connected and on each of its connected components the action is bound by the LHS.

It is an easy exercise to show that the LHS is invariant under a small deformation of the projector. In fact, when θ is irrational, two projectors lie in the same connected component iff they have the same trace [18]. Furthermore, if the normalized trace of

P is $p + q\theta$, then

$$\frac{\epsilon_{\mu\nu}}{(2\pi R)^2} \int P\partial_\mu P\partial_\nu P = 2i\pi q\,, \tag{5.5}$$

so that q is an analogue of a two-dimensional instanton number. Indeed, it has been obtained in [33] as a topological bound in the study of a non-commutative generalization of a non-linear σ-model. It is part of a general theory that encompasses both this model and the ordinary non-linear field theory with values in S^2. Because S^2 is homeomorphic to the space of rank one projectors in $M_2(\mathbb{C})$, it is easy to write the kinetic term of the standard non-linear σ model and (5.5) is nothing but the winding number of the corresponding map from S^2 into itself. This means that it measures the homotopy class of this map, i.e. it is an element of $\pi_2(S^2)$.

In the context of non-commutative geometry, the rôle of the homotopy groups is played by the K-theory of the algebra, (i.e. classes of projectors and unitary elements of matrix algebra over the algebra of coordinates). They provide, when suitably differentiated and integrated, quantities that are stable under small deformations. In a more abstract language, this is formulated through the pairing of the cyclic cohomology of the algebra (i.e. all the possible ways to differentiate and integrate in a "suitable way" elements of the algebra) with its K-theory (i.e. non-commutative analogues of homotopy classes of vector bundles and gauge transformation). This theory is fully developed in the treatise [35] and the recent review [36].

Turning back to the problem of minimizing (5.1), it follows from (5.4) that the bound will be saturated iff $\partial P\, P = 0$ (antiself-duality equation) or $\overline{\partial} P\, P = 0$ (self-duality equation). Because of the non-linear constraint arising form the fact that P must be a projector, these equations are not easy to solve.

If P is a self-dual projector, then $1 - P$ is an antiself-dual and vice-versa, so that it is sufficient to look only for the former. Fortunately, this non-linear problem can be turned into a linear one using the bimodule technique.

Using the notation of section 3, we recall that it allows to construct a projector in $\mathcal{A}$ provided we have an element $\psi \in \mathcal{M}$ such that $(\psi, \psi)_{\mathcal{B}} = 1$. For simplicity, we restrict our discussion to the homotopy class of the Powers-Rieffel projector, so that we know that all projectors in this class are obtained through unit vector in the bimodule $\mathcal{N} = \mathcal{S}(\mathbb{R})$. It follows that the resulting projector will satisfy

$$\frac{1}{(2\pi R)^2} \int P = \theta\,, \qquad \text{and} \qquad \frac{\epsilon_{\mu\nu}}{(2\pi R)^2} \int P\partial_\mu P\partial_\nu P = 2i\pi\,. \tag{5.6}$$

Note that the map from from unit vectors to projectors is not one to one; two unit vectors yield the same projector iff they differ by a gauge transformation. Indeed, if ψ and χ are unit vectors such that

$$P = (\psi, \psi)_{\mathcal{A}} = (\chi, \chi)_{\mathcal{A}} \tag{5.7}$$

then

$$\psi = (\psi, \chi)_{\mathcal{B}}\, \psi\,, \qquad \text{and} \qquad \chi = (\chi, \psi)_{\mathcal{B}}\, \chi \tag{5.8}$$

and the gauge transformation is $u = (\chi, \psi)_{\mathcal{B}}$ which belongs to the gauge group (i.e. the group of unitary elements of $\mathcal{B}$).

A natural way to construct a unit vector ψ is to start with an arbitrary element $\chi \in \mathcal{E}$ whose norm $(\chi, \chi)_{\mathcal{B}}$ is invertible. Then standard mathematical techniques (holomorphic functional calculus, for instance) allow us to define the square root of $(\chi, \chi)_{\mathcal{B}}$ so that

$$\psi = ((\chi, \chi)_{\mathcal{B}})^{-1/2}\, \chi \tag{5.9}$$

is a unit vector.

The main difficulty of this approach is to determine whether the norm (χ, χ) is invertible or not. For instance, if $\theta > 1$, we know that $(\chi, \chi)_{\mathcal{B}}$ is not invertible, otherwise we would have constructed a projector of trace $\theta > 1$ in $\mathcal{A}_\theta$, which is impossible. On the other hand, for $\theta = 1/n$, $\mathcal{B}$ is commutative and $(\chi, \chi)_{\mathcal{B}}$ is invertible iff this function does not vanish.

In trading the projectors for the vectors $\chi \in \mathcal{E}$, we have introduced spurious gauge degrees of freedom. In fact, two such vectors yield the same projector iff they differ by a complex (not necessarily unitary) gauge transformation. By definition such an element belongs to the group of invertible elements of $\mathcal{B}$. Therefore, we have to identify the vectors that yield self-dual projectors but differ only by a complex gauge transformation.

Let us introduce the complex covariant derivatives

$$\nabla = \nabla_1 - i\nabla_2 \tag{5.10}$$

associated to the covariant derivatives ∇_1 and ∇_2 introduced in section 3.

Now we have all the tools to solve the self-duality equation. In fact the projector P associated to a vector χ of invertible norm satisfies the equation

$$\bar{\partial} P P = 0 \tag{5.11}$$

iff there is $\rho \in \mathcal{B}$ such that

$$\nabla \chi = \rho \chi\,. \tag{5.12}$$

This statement follows from an explicit computation of $\overline{\partial} P\, P$ in terms of χ.

Consequently, our problem has been reduced to the following linear problem. Given any $\rho \in \mathcal{B}$, we have to find all $\chi \in \mathcal{E}$ such that

$$\nabla \chi = \rho \chi\,. \tag{5.13}$$

This is a linear equation in ψ and all the powerful methods of linear functional analysis can be applied to this problem.

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The complex gauge group acts on ρ as

$$\rho \longrightarrow g\rho g^{-1} + g\partial g^{-1} \tag{5.14}$$

so that ρ is nothing but a complex gauge field. Moreover, if $\lambda \in \mathcal{B}$ can be deduced from ρ by a complex gauge transformation

$$\lambda = g^{-1}\rho g + g^{-1}\overline{\partial} g \tag{5.15}$$

then χ satisfies the linear problem associated to ρ iff $g\chi$ satisfies the linear problem associated to λ. This means that we have to solve the linear problem only on the orbit of the action of the complex gauge group on the complex connection. In other words, if we find a subset of $\mathcal{B}$ that intersects each orbit at least once, it is sufficient to solve linear problem for those values of ρ.

The general problem of the study of these orbit spaces is a rather difficult problem which is tantamount to the study of the moduli spaces of complex vector bundles over the non-commutative torus. In the special case we are considering here, one can show that constants intersect each orbit at least once, thus we have to solve the equation

$$\nabla f = \frac{\lambda}{2\pi R} f\,, \tag{5.16}$$

with $\lambda \in \mathbb{C}$. This is very easy because (5.16) is just the differential equation

$$f^{'}(t) + \frac{2\pi t}{\theta} f(t) - \lambda i f(t) = 0\,, \tag{5.17}$$

whose solution is the gaussian

$$f_\lambda(t) = A e^{-\frac{\pi t^2}{\theta} + \lambda i t} \tag{5.18}$$

A being an arbitrary constant, which is absolutely inessential since it cancels in the expression for the projector. Projector based on this function has been for the first time constructed by Boca [34]. The self-duality property was recognized in [33].[2] It has been also studied in [10].

Besides, one can show that two values of λ differ by a gauge transformation iff they belong to the same class in $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, where τ is the modular parameter of our initial torus [33]. Here we have set $\tau = i$, but it is an easy exercise to work with a general value of τ and observe the covariance under transformation in $\mathrm{SL}_2(\tau)$. Therefore, the torus parametrizes all instantons in the homotopy class of the Powers-Rieffel projector.

Note added. After this work has been completed we have received a preprint [37] which also deals with solitons on non-commutative tori.

[2]Note that the Boca's projector shares this self-duality property with its Moyal plane counterpart, the simplest GMS soliton. Indeed, in the operator correspondence $\bar{\partial} PP \sim [a, |0\rangle\langle 0|]|0\rangle\langle 0| = 0$.

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A. Moyal plane limit of Boca's projector

The Powers-Rieffel projector [18] discussed in the context of tachyon condensation in [7]–[10] is fully legitimate one in the sense that one can find exact solitons based on this projector. Nonetheless one may wish to have projector which does reduce to the nice GMS solitons [3] in the large torus limit keeping θ_{Moyal} fixed. The Powers-Rieffel projector is also not suitable for orbifolding [10].

A projector which overcomes these difficulties was constructed by Boca [34] using the bimodule technique starting from the gaussian function $f = e^{-\pi t^2/\theta}$. He has proved for certain range of θ that $b = \langle f, f\rangle_{\mathcal{B}}$ is invertible. Then one can easily check that

$$P_\theta = \langle b^{-1/2} f, b^{-1/2} f\rangle_{\mathcal{A}} \tag{A.1}$$

is a projector. For special values of $\theta = 1/q, q \in \mathbb{Z}$ the projector can be expressed explicitly in terms of Jacobi's theta functions. For example for q even one has

$$P_{1/q} = \frac{\sum_{r,s=0}^{q-1} e^{-\pi i r s/q} \vartheta^{(q)}_{s/q,r/2}(U_2, iq/2) \vartheta^{(q)}_{r/q,s/2}(U_1, iq/2)}{q\vartheta(U_2^q, iq/2)\vartheta(U_1^q, iq/2)}\,, \tag{A.2}$$

where

$$\begin{aligned} \vartheta^{(N)}_{a/N,b}(U,\tau) &= \sum_m e^{\pi i \tau (m+\frac{a}{N})^2 + 2\pi i (m+\frac{a}{N})b} U^{Nm+a}\,, \\ \vartheta(U,\tau) &= \sum_m e^{\pi i \tau m^2} U^m\,. \end{aligned} \tag{A.3}$$

Note that the above formula (A.2) makes sense as it stands since the denominator turns out to be central element in the algebra. One can translate back this expression in the language of ordinary functions and the Moyal star product. With some effort one can check that in the large torus limit keeping θ_{Moyal} fixed (which is equivalent to $q \to \infty$), the above projector goes to the basic GMS soliton

$$P = |0\rangle\langle 0| \simeq 2e^{-\frac{x_1^2+x_2^2}{|\theta_{\mathrm{Moyal}}|}}\,. \tag{A.4}$$

In this limit the theta functions in the denominator do not contribute. In the numerator the sums over r and s factorize, after some rearrangement one can replace them again by Jacobi's theta function, use its duality property and obtain the result with all the factors right.

An alternative way is to start with the formula for $b^{-1/2}f$ obtained in [10]. For small θ it simply reduces to

$$b^{-1/2}f \sim \sqrt[4]{\frac{2}{\theta}}f\,. \tag{A.5}$$

From the formula (3.19) we first calculate

$$\begin{aligned}\sqrt{\frac{2}{\theta}}\langle f,f\rangle_{\mathcal{A}} &= \theta\sum_{m,n} e^{-\frac{\pi\theta}{2}(m^2-2imn+n^2)}e^{im\frac{y}{R}} * e^{in\frac{x}{R}}\\ &= \theta\,\vartheta\left(\frac{x}{2\pi R},\frac{i\theta}{2}\right)\vartheta\left(\frac{y}{2\pi R},\frac{i\theta}{2}\right).\end{aligned} \tag{A.6}$$

Note that in the second line the product between the theta functions is the ordinary commutative one. The theta function is the basic one defined by

$$\vartheta(\nu,\tau) = \sum_m e^{\pi i\tau m^2+2\pi i m\nu} \tag{A.7}$$

with the familiar modular transformation property

$$\vartheta\left(\frac{\nu}{\tau},-\frac{1}{\tau}\right) = (-i\tau)^{1/2}e^{\pi i\nu^2\tau}\vartheta(\nu,\tau)\,. \tag{A.8}$$

From that simply follows the asymptotic behavior

$$\langle b^{-1/2}f, b^{-1/2}f\rangle_{\mathcal{A}} \sim 2e^{-\frac{x_1^2+x_2^2}{|\theta_{\mathrm{Moyal}}|}} \tag{A.9}$$

which is just as for the GMS soliton.

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